MOTIVES FOR PERFECT PAC FIELDS WITH PRO-CYCLIC GALOIS GROUP

IMMANUEL HALUPCZOK

ABSTRACT. Denef and Loeser defined a map from the Grothendieck ring of sets definable in pseudo-finite fields to the Grothendieck ring of Chow motives, thus enabling to apply any cohomological invariant to these sets. We generalize this to perfect, pseudo algebraically closed fields with pro-cyclic Galois group.

In addition, we define some maps between different Grothendieck rings of definable sets which provide additional information, not contained in the associated motive. In particular we infer that the map of Denef-Loeser is not injective.

CONTENTS

1991 *Mathematics Subject Classification.* 03C10, 03C60, 03C98, 12L12, 12E30, 12F10, 14G15, 14G27.

Key words and phrases. pseudo-finite fields, pseudo algebraically closed fields, motives.

The author was supported by the Agence National de la Recherche (contract ANR-06-BLAN-0183-01).

1. Preliminaries

1.1. **Introduction.** To understand definable sets of a theory, it is helpful to have invariants with nice properties. For a fixed pseudo-finite field K, there are two well-known invariants of definable sets: the dimension (see [3]), and the measure (see [2]).

In a slightly different setting, Denef and Loeser constructed a much stronger invariant: they do not fix a pseudo-finite field; instead they consider definable sets in the theory of all pseudo-finite fields of characteristic zero. To each such set X they associate an element $\chi_c(X)$ of the Grothendieck ring of Chow motives (see [4], [5]). In particular, this implies that all the usual cohomological invariants (like Euler characteristic, Hodge polynomial) are now applicable to arbitrary definable sets.

The dimension defined in [3] exists for a much larger class of fields and in [8], Hrushovski asked whether one can also generalize the measure. This question has been answered in [7]: it is indeed possible to define a measure for any perfect, pseudo algebraically closed (PAC) field with pro-cyclic Galois group. A natural question is now: can the work of Denef-Loeser also be generalized to this setting? More precisely, fix a torsion-free pro-cyclic group Gal and consider the theory of perfect PAC fields with absolute Galois group Gal. Then to any definable set X in that theory we would like to associate a virtual motive $\chi_c(X)$. The first goal of this article is to do this (Theorem 1.1).

One reason this result seems interesting to me is the following: the map χ_c exists for pseudo-finite fields (by Denef-Loeser) and for algebraically closed fields (by quantifier elimination). The case of general pro-cyclic Galois groups is a common generalization of both and thus a kind of interpolation.

Comparing those maps χ_c for different Galois groups, one gets the feeling that they are closely related. Indeed, given an inclusion of Galois groups $Gal_2 \subset Gal_1$, we will prove (in Theorem 1.3) the existence of a map θ from the definable sets for Gal_2 to the definable sets for Gal_1 which is compatible with the different maps χ_c.

These maps θ turn out to be interesting in themselves. An open question was whether the map χ_c is injective for pseudo-finite fields. We will show (Proposition 1.4) that it is not, by giving an example of two definable sets with the same image under χ_c but with different images under one of those maps θ. This also means that at least in this case, the maps θ can be used to get information which one loses by applying χ_c.

We have one more result. In [5], the map χ_c is defined by enumerating certain properties and then existence and uniqueness of such a map is proven. We are able to weaken the conditions needed for uniqueness in the case of pseudo-finite fields. Unfortunately however, we do not get any sensible uniqueness conditions for other pro-cyclic Galois groups.

1.2. **The results in detail.** Let us fix some notation once and for all.

By a "group homomorphism" we will always mean a continuous group homomorphism if there are pro-finite groups involved.

We fix a field of parameters k and a group Gal which will serve as Galois group. Sometimes, we will require k to be of characteristic zero. Gal will always be a pro-cyclic group such that there do exist perfect PAC fields having Gal as absolute

Galois group. This is the case if and only if *Gal* is torsion-free, or equivalently, if it is of the form $\prod_{p\in P}\mathbb{Z}_p$, where P is any set of primes.

The theory we will be working in will be the theory of perfect PAC fields with absolute Galois group *Gal* which contain k. We will denote this theory by $T_{Gal,k}$. Models of $T_{Gal,k}$ will be denoted by K; the algebraic closure of a field K will be denoted by $\tilde{K}$. By "definable" we always mean 0-definable. (But k is part of the language.)

By "variety", we mean a separated, reduced scheme of finite type. If not stated otherwise, all our varieties will be over k.[1]

We will use the notion "definable set" even when there is no model around: by a "definable set (in $T_{Gal,k}$)", we mean a formula up to equivalence modulo $T_{Gal,k}$. In addition, we will permit ourselves to speak about "definable subsets of (arbitrary) varieties". For affine embedded varieties, it is clear what this should mean. In general, any definable decomposition of a variety V into affine embedded ones yields the same notion of definable subsets of V (cf. "definable sub-assignments" in [4]).

We will use the usual definitions of the following Grothendieck rings (see e.g. [4] or [5]): the Grothendieck ring of varieties $\mathrm{K}_0(\mathrm{Var}_k)$, the Grothendieck ring of (Chow) motives $\mathrm{K}_0(\mathrm{Mot}_k)$ and the Grothendieck ring $\mathrm{K}_0(T_{Gal,k})$ of the theory $T_{Gal,k}$. Moreover, we will often need to tensor the Grothendieck ring of motives with $\mathbb{Q}$; we denote this by $\mathrm{K}_0(\mathrm{Mot}_k)_\mathbb{Q} := \mathrm{K}_0(\mathrm{Mot}_k)\otimes_\mathbb{Z}\mathbb{Q}$.

Now let us state the generalization of the theorem of Denef-Loeser. For the definition of "Galois cover" and "$X(V \overset{G}{\twoheadrightarrow} W, \{1\})$", see Section 2.

Theorem 1.1. *Suppose $Gal = \prod_{p\in P}\mathbb{Z}_p$ (where P is any set of primes) is a torsion-free pro-cyclic group and k is a field of characteristic zero. Then there exists a (canonical) ring homomorphism $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_\mathbb{Q}$ extending the usual homomorphism $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k)\to \mathrm{K}_0(\mathrm{Mot}_k)$ with the following property: if $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover such that all prime factors of $|G|$ lie in P, then*

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)\,.$$

If $Gal = \hat{\mathbb{Z}}$, then a ring homomorphism with these properties is unique.

As already mentioned, our condition $(*)$ needed for uniqueness in the pseudo-finite case is weaker than the one of Denef-Loeser (Theorem 6.4.1 of [5]).

If $Gal \neq \hat{\mathbb{Z}}$, we can not prove that condition $(*)$ is strong enough to define χ_c uniquely, and we do not have any good replacement for $(*)$. Nevertheless, we will sometimes speak of *the* map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_\mathbb{Q}$ and mean the one defined in Section 3.2 (after Lemma 3.3).

The map χ_c does not really depend on the base field k: if we have a second field k' containing k, then there are canonical ring homomorphisms $\mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(T_{Gal,k'})$ and $\mathrm{K}_0(\mathrm{Mot}_k)_\mathbb{Q} \to \mathrm{K}_0(\mathrm{Mot}_{k'})_\mathbb{Q}$, which we will both denote by $\otimes_k k'$. The map χ_c is compatible with these homomorphisms:

Proposition 1.2. *In the setting just described we have, for any definable set X of $T_{Gal,k}$, $\chi_c(X\otimes_k k') = \chi_c(X)\otimes_k k'$.*

[1] We will try to limit our notation such that readers not so familiar with the language of schemes can use a more naive definition of varieties. For those readers: our varieties are not supposed to be irreducible.

We will not write down the proof of this, as it is exactly the same as in the pseudo-finite case; see [4], the paragraph before Lemma 3.4.1, or [9], Proposition 8.9.

The next theorem is the one putting the Grothendieck rings of theories corresponding to different Galois groups into relation.

Theorem 1.3. *Suppose Gal_1 and Gal_2 are two torsion-free pro-cyclic groups, $\iota\colon Gal_2 \hookrightarrow Gal_1$ is an injective map, and k is any field (not necessarily of characteristic zero). Denote the theories $T_{Gal_i,k}$ by T_i for $i = 1, 2$. Then the following defines a ring homomorphism $\theta_\iota\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$: Suppose K_1 is a model of T_1. Then the fixed field $K_2 := \tilde{K}_1^{\iota(Gal_2)}$ is a model of T_2 containing K_1. For any $X_2 \subset \mathbb{A}^n$ definable in T_2, we define $\theta_\iota(X_2)(K_1) := X_2(K_2) \cap K_1^n$.*

Using this theorem, one can reduce the existence of χ_c for arbitrary torsion-free pro-cyclic groups Gal to the case $Gal = \hat{\mathbb{Z}}$ (which has been treated by Denef-Loeser): apply Theorem 1.3 to $\iota\colon Gal \hookrightarrow \hat{\mathbb{Z}}$, where ι maps Gal to the appropriate factor $\prod_{p\in P} \mathbb{Z}_p$ of $\hat{\mathbb{Z}}$ (such that $\hat{\mathbb{Z}}/Gal$ is torsion-free). Then define χ_c as the composition $\hat{\chi}_c \circ \theta_\iota$, where $\hat{\chi}_c\colon \mathrm{K}_0(T_{\hat{\mathbb{Z}},k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is the known map in the pseudo-finite case. Verification of the properties of χ_c is not very difficult using the explicit computations done in the proof of Theorem 1.3.

So in principle, we are done with the existence part of Theorem 1.1 (provided we can prove Theorem 1.3). On the other hand, one has the feeling that it should also be possible to construct the map χ_c directly for any group Gal. We will do this in Section 3.2, but as our construction closely follows the construction in [9], we will go into details only in places where our generalization requires some modifications.

Another interesting application of Theorem 1.3 is the case $Gal_1 = Gal_2 = Gal$, but with a non-trivial injection $\iota\colon Gal \hookrightarrow Gal$. One thus gets endomorphisms of the ring $\mathrm{K}_0(T_{Gal,k})$, which might reveal a lot of information about its structure. Indeed using such endomorphisms we will construct a whole family of pairs of definable sets X_1 and X_2 such that $\chi_c(X_1) = \chi_c(X_2)$ but $\chi_c(\theta(X_1)) \neq \chi_c(\theta(X_2))$, thereby proving:

Proposition 1.4. *Let k be a field of characteristic zero and let Gal be a non-trivial torsion-free pro-cyclic group. Then the map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is not injective.*

The remainder of the article is organized as follows. In Section 2, we state the main tool to get hold of arbitrary definable sets, namely quantifier elimination to Galois formulas. Before that, we introduce the necessary notation: Galois covers, a generalized Artin symbol and Galois stratifications. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the maps θ_ι: we prove Theorem 1.3 and Proposition 1.4, and moreover, we check that the maps χ_c of Theorem 1.1 for different Galois groups are compatible with suitable maps θ_ι (Proposition 4.1). Finally Section 5 lists some open problems.

2. Galois stratifications and quantifier elimination

A standard technique to get hold of definable sets of perfect PAC fields with not-too-large Galois group is the quantifier elimination to Galois formulas. In this section, we define the necessary objects and then, in Section 2.4, state this quantifier elimination result in the version of Fried-Jarden [6].

2.1. Galois covers.

Definition 2.1.
(1) *A* Galois cover *consists of two integral and normal varieties V and W (over some fixed field k) and a finite étale map $f\colon V \to W$ such that for $G := \mathrm{Aut}_W(V)^{\mathrm{opp}}$, we have canonically $W \cong V/G$ (where G acts from the right on V). We denote a Galois cover by $f\colon V \overset{G}{\twoheadrightarrow} W$ and call G the* group *of that cover. The action of G on V will be denoted by $v.g$ (for $v \in V$, $g \in G$).*
(2) *We say that a Galois cover $f'\colon V' \overset{G'}{\twoheadrightarrow} W$ is a* refinement *of $f\colon V \overset{G}{\twoheadrightarrow} W$, if there is a finite étale map $g\colon V' \twoheadrightarrow V$ such that $f' = f \circ g$.*
(3) *If W'' is a locally closed subset of W and V'' is a connected component of $f^{-1}(W'')$, then we call $V'' \overset{G''}{\twoheadrightarrow} W''$ the* restriction *of $V \overset{G}{\twoheadrightarrow} W$ to W'', where $G'' := \mathrm{Aut}_{W''}(V'')^{\mathrm{opp}}$.*

Remark 2.2.
(1) *If $f'\colon V' \overset{G'}{\twoheadrightarrow} W$ is a refinement of $f\colon V \overset{G}{\twoheadrightarrow} W$, then we have a canonical surjection $\pi\colon G' \twoheadrightarrow G$.*
(2) *If $V'' \overset{G''}{\twoheadrightarrow} W''$ is a restriction of $V \overset{G}{\twoheadrightarrow} W$, then we have a canonical injection $G'' \hookrightarrow G$. Different choices of the connected component of $f^{-1}(W'')$ yield isomorphic restricted Galois covers.*

2.2. Artin symbols and colorings.

Using a Galois cover $V \overset{G}{\twoheadrightarrow} W$, we would like to decompose W into subsets according to the Artin symbol of the elements. However, the usual definition of Artin symbol needs a canonical generator of the Galois group *Gal* (usually the Frobenius of a finite field); the Artin symbol is then the image of the generator under a certain map $\rho\colon\ Gal \to G$ (which is unique only up to conjugation by G). If one does not have such a canonical generator, then one still can consider the image of ρ. This is what one uses as Artin symbol in our case (see [6]).

Definition 2.3 (and Lemma)**.** *Suppose $f\colon V \overset{G}{\twoheadrightarrow} W$ is a Galois cover over k and K is a field containing k.*
(1) *Suppose $v \in V(\tilde{K})$ such that $f(v) \in W(K)$. Then there is a unique group homomorphism $\rho\colon \mathrm{Gal}(\tilde{K}/K) \to G$ satisfying $\sigma(v) = v.\rho(\sigma)$ for any $\sigma \in \mathrm{Gal}(\tilde{K}/K)$. The* decomposition group $\mathrm{Dec}(v) := \mathrm{im}\,\rho \subset G$ *of v is the image of that homomorphism.*
(2) *For $w \in W(K)$, let the* Artin Symbol $\mathrm{Ar}(w)$ *of w be the set $\{\mathrm{Dec}(v) \mid v \in V(\tilde{K}), f(v) = w\}$ of decomposition groups of all preimages of w.*

$\mathrm{Ar}(w)$ consists exactly of one conjugacy class of subgroups of G, and these subgroups are isomorphic to a quotient of the absolute Galois group $\mathrm{Gal}(\tilde{K}/K)$ of the field.

If K is a model of our theory T, then the quotients of $\mathrm{Gal}(\tilde{K}/K) = Gal$ are just the cyclic groups Q such that all prime factors of $|Q|$ lie in P (where P is the set of primes such that $Gal = \prod_{p\in P} \mathbb{Z}_p$). We introduce some notation for this:

Definition 2.4. *Given a finite group G, we will call those subgroups of G which are isomorphic to a quotient of Gal the* permitted subgroups*. We denote the set of all permitted subgroups of G by* $\mathrm{Psub}(G)$*. If Q is a finite cyclic group, then we denote by* $\mathrm{Ppart}(Q)$ *the "permitted part of Q", i.e. the biggest permitted subgroup of Q.*

The interest of $\mathrm{Ppart}(Q)$ is the following. We will sometimes identify $Gal = \prod_{p\in P} \mathbb{Z}_p$ with the corresponding factor of $\hat{\mathbb{Z}}$ and consider homomorphisms $\rho\colon \hat{\mathbb{Z}} \to G$. Then the image of Gal in G is just $\rho(Gal) = \mathrm{Ppart}(\operatorname{im}\rho)$.

Given a Galois cover $V \overset{G}{\twoheadrightarrow} W$, we now define subsets of W using the Artin symbol:

Definition 2.5. (1) *A* coloring *of a Galois cover $V \overset{G}{\twoheadrightarrow} W$ is a subset C of the permitted subgroups of G which is closed under conjugation. A Galois cover together with a coloring is called a* colored Galois cover.

(2) *Given a colored Galois cover $(V \overset{G}{\twoheadrightarrow} W, C)$ and a model $K \models T$, we define the set $X(V \overset{G}{\twoheadrightarrow} W, C)(K) := \{w \in W(K) \mid \mathrm{Ar}(w) \subset C\}$.*

Note that $X(V \overset{G}{\twoheadrightarrow} W, C)$ is definable, i.e. there is a formula ϕ such that for any model $K \models T$ we have $\phi(K) = X(V \overset{G}{\twoheadrightarrow} W, C)(K)$.

Remark 2.6. (1) *If $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $V' \overset{G'}{\twoheadrightarrow} W$ is a refinement with canonical map $\pi\colon G' \twoheadrightarrow G$, then we can also refine the coloring: by setting $C' := \{Q \in \mathrm{Psub}(G') \mid \pi(Q) \in C\}$, we get $X(V' \overset{G'}{\twoheadrightarrow} W, C') = X(V \overset{G}{\twoheadrightarrow} W, C)$.*

(2) *Similarly if $V'' \overset{G''}{\twoheadrightarrow} W''$ is a restriction of $f\colon V \overset{G}{\twoheadrightarrow} W$: in that case, set $C'' := \{Q \in C \mid Q \subset G''\}$. Then we get $X(V'' \overset{G''}{\twoheadrightarrow} W'', C'') = X(V \overset{G}{\twoheadrightarrow} W, C) \cap W''$.*

2.3. **Galois stratifications.**

Definition 2.7. *A* Galois stratification $\mathcal{A}$ *of a variety W is a finite family $(f_i\colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)_{i\in I}$ of colored Galois covers where the W_i form a partition of W into locally closed sub-varieties. We shall say that $\mathcal{A}$* defines *the following subset $\mathcal{A}(K) \subset W(K)$, where $K \models T$ is a model:*

$$\mathcal{A}(K) := \bigcup_{i\in I} X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)(K)$$

The data of a Galois stratification denoted by $\mathcal{A}$ will always be denoted by V_i, W_i, G_i, C_i, and analogously with primes for $\mathcal{A}'$, $\mathcal{A}''$, etc. This will not always be explicitely mentioned.

Definition 2.8. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications. We say that $\mathcal{A}'$ is a* refinement *of $\mathcal{A}$, if:*

- *Each W_i is a union $\bigcup_{j\in J_i} W'_j$ for some $J_i \subset I'$.*
- *For each $i \in I$ and each $j \in J_i$, the Galois cover $V'_j \overset{G'_j}{\twoheadrightarrow} W'_j$ is a refinement of the restriction of the Galois cover $V_i \overset{G_i}{\twoheadrightarrow} W_i$ to the set W'_j.*
- *C'_j is constructed out of C_i as described in Remark 2.6, such that $X(V'_j \overset{G'_j}{\twoheadrightarrow} W'_j, C'_j) = X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i) \cap W'_j$.*

By the third condition, $\mathcal{A}$ and $\mathcal{A}'$ define the same set.

One reason for Galois stratifications being handy to use is the following general lemma:

Lemma 2.9. *If $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications, then there exist refinements $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ of $\mathcal{A}$ resp. $\mathcal{A}'$ which differ only in the colorings.*

2.4. **Quantifier elimination to Galois stratifications.** We now state the version of quantifier elimination which we will use. It is given in [6], Proposition 30.5.2. Applied to our situation, that proposition reads:

Lemma 2.10. *Suppose Gal is a torsion-free pro-cyclic group and k is any field. Then each definable set X of $T_{Gal,k}$ is already definable by a Galois stratification $\mathcal{A}$ (over k), i.e. for any $K \models T_{Gal,k}$, we have $X(K) = \mathcal{A}(K)$.*

Note that Proposition 30.5.2 of [6] requires that K is what Fried-Jarden call a "prefect Frobenius field"; this is indeed the case for any model of $T_{Gal,k}$.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 without using Theorem 1.3: we construct the map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$, check its properties, and prove uniqueness in the case $Gal = \hat{\mathbb{Z}}$. For the whole section, we fix a torsion-free pro-cyclic group Gal and a field k of characteristic zero. We also fix the theory $T := T_{Gal,k}$ we will be working in.

3.1. **Some preliminary lemmas.** We will need the following basic property of the generalized Artin symbol.

Lemma 3.1. *Suppose we have the following commutative diagram of varieties over k, where the maps $f_1\colon V \to W_1$ and $f_2\colon V \to W_2$ are Galois covers with groups G_1 and G_2, respectively. We have naturally $G_1 \subset G_2$.*

$$\begin{tikzcd} V \arrow[d, two heads, "f_1"'] \arrow[dr, two heads, "f_2"] & \\ W_1 \arrow[r, two heads, "\phi"'] & W_2 \end{tikzcd}$$

Suppose additionally that C_1 is a conjugacy class of subgroups of G_1 and $C_2 := C_1^{G_2}$ is the induced conjugacy class of subgroups of G_2. Then for any field $K \supset k$, the image under ϕ of $X_1(K) := \{w_1 \in W_1(K) \mid \mathrm{Ar}(w_1) = C_1\}$ is $X_2(K) := \{w_2 \in W_2(K) \mid \mathrm{Ar}(w_2) = C_2\}$. Moreover, the size of the fibers of the induced map $X_1(K) \to X_2(K)$ is $\frac{|G_2|\cdot|C_1|}{|C_2|\cdot|G_1|}$.

The following lemma can be seen as a qualitative version of Chebotarev's density theorem, where the finite fields have been replaced by models of our theory. However, the proof is much easier than the one of the usual density theorem.

Lemma 3.2. *Suppose $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover with $C \neq \emptyset$.*

1. *There exists a model $K \models T$ such that $X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ contains an element which is generic over k.*
2. *If $K \models T$ is a model such that W is irreducible over K and $X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ is not empty, then $X(V \overset{G}{\twoheadrightarrow} W, C)(K)$ is already dense in $W(K)$.*

Part (1) follows from Theorem 23.1.1 of [6]; part (2) follows from Proposition 24.1.4 of [6]. For details, see Corollary 2.9 and Lemma 2.10 of [9]: the proofs there (which are for pseudo-finite fields) directly generalize to models of T.

3.2. **Existence of χ_c.** The proof of the existence of the map χ_c of Theorem 1.1 consists of three parts:

(1) Define a virtual motive associated to a colored Galois cover.
(2) Generalize this definition to Galois stratifications and verify that the virtual motive defined in this way only depends on the set defined by the stratification.

    Using the quantifier elimination result Lemma 2.10, we thus get a map χ_c from the definable sets to the virtual motives.
(3) Check that this map χ_c has all the required properties: that it is invariant under definable bijections and compatible with disjoint union and products (so it defines a ring homomorphism $\mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$) and that it satisfies condition $(*)$ of Theorem 1.1.

(1) To associate a virtual motive to a colored Galois cover $(V \overset{G}{\twoheadrightarrow} W, C)$, one first associates a central function $\alpha_C \colon G \to \mathbb{Q}$ to the coloring, and then one uses a result from [1] to turn this into a virtual motive.

More precisely, let $C(G, \mathbb{Q})$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-central functions, i.e. the space of functions $\alpha \colon G \to \mathbb{Q}$ such that $\alpha(g) = \alpha(g')$ whenever $g, g' \in G$ generate conjugate subgroups of G. The following result essentially follows from Theorem 6.1 of [1]; see [4] or [9] for more details.

Lemma 3.3. *There exists a (unique) map χ_c which associates to each finite group G, each G-variety V and each $\mathbb{Q}$-central function $\alpha \in C(G, \mathbb{Q})$ a virtual motive $\chi_c(G \looparrowright V, \alpha) \in \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ and which has the following properties:*

(1) *For any fixed G and α, the induced map from the Grothendieck ring of G-varieties to $\mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is a group homomorphism.*
(2) *For any fixed G and V, the induced map $C(G, \mathbb{Q}) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear.*
(3) *If α_{reg} is the character of the regular representation of G, then $\chi_c(G \looparrowright V, \alpha_{\mathrm{reg}}) = \chi_c(V)$.*
(4) *Suppose G is a group acting on a variety V, H is a normal subgroup, $\pi \colon G \twoheadrightarrow G/H$ is the projection, and $\alpha \in C(G/H, \mathbb{Q})$ is a $\mathbb{Q}$-central function. Then*
$$\chi_c(G/H \looparrowright V/H, \alpha) = \chi_c(G \looparrowright V, \alpha \circ \pi)\,.$$
(5) *Suppose G is a group acting on a variety V, $H \subset G$ is any subgroup, and $\alpha \in C(H, \mathbb{Q})$ is a $\mathbb{Q}$-central function. Then*
$$\chi_c(G \looparrowright V, \mathrm{Ind}_H^G \alpha) = \chi_c(H \looparrowright V, \alpha)\,.$$

(Several other properties are omitted. See e.g. [4]*, Theorem 3.1.1 and Proposition 3.1.2 or* [9]*, Section 7.)*

Using this, one defines
$$\chi_c(V \overset{G}{\twoheadrightarrow} W, C) := \chi_c(G \looparrowright V, \alpha_C)\,,$$
where α_C still has to be defined.

In the case of pseudo-finite fields, one defines α_C to be 1 on the set $\{g \in G \mid \langle g \rangle \in C\}$ and 0 elsewhere. Just copying this definition does not work when the Galois group is not $\hat{\mathbb{Z}}$. The reason is that the meaning of "$Q \in C$" is different when the Galois group of the field is not $\hat{\mathbb{Z}}$. For example, "$\{1\} \in C$" means "just a little

part of W" when $Gal = \hat{\mathbb{Z}}$, whereas when Gal is trivial, it means "the whole of W".

To get a working definition for α_C in the non-$\hat{\mathbb{Z}}$-case, one has to recall that the Artin symbol is the image of a certain map $\rho\colon\ Gal \to G$. Then one views Gal as a subgroup of $\hat{\mathbb{Z}}$ and considers extensions of ρ to $\hat{\mathbb{Z}}$, as described in the remark after Definition 2.4. In this way one naturally gets the following definition, which will turn out to work:

$$\alpha_C(g) := \begin{cases} 1 & \text{if } \operatorname{Ppart}(\langle g\rangle) \in C \\ 0 & \text{otherwise}. \end{cases}$$

(2) We generalize the map χ_c from colored Galois covers to Galois stratifications in the obvious way:

$$\chi_c(\mathcal{A}) := \sum_{i\in I} \chi_c(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)\,.$$

Now suppose that two Galois stratifications $\mathcal{A}$ and $\mathcal{A}'$ define the same set. To check that the associated motives $\chi_c(\mathcal{A})$ and $\chi_c(\mathcal{A}')$ are the same, we use Lemma 2.9. It is enough to show that (a) refining a stratification does not change the motive and that (b) if two colorings of a Galois cover define the same set, then these colorings are equal. Refinement of stratifications decomposes into two parts: (a1) refining the underlying sets W_i and (a2) refining the Galois covers.

(a1) is straight forward.

(a2) We have to show that $\chi_c(V \overset{G}{\twoheadrightarrow} W, C) = \chi_c(V' \overset{G'}{\twoheadrightarrow} W, C')$ where $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $(V' \overset{G'}{\twoheadrightarrow} W, C')$ is a refinement. By Lemma 3.3 (4), it is enough to check that $\alpha_{C'} = \alpha_C \circ \pi$, where $\pi\colon G' \twoheadrightarrow G$ is the canonical map. But indeed we have, for any $g' \in G'$:

$$\begin{aligned} \alpha_{C'}(g') = 1 &\iff \operatorname{Ppart}(\langle g'\rangle) \in C' \iff \pi(\operatorname{Ppart}(\langle g'\rangle)) \in C \\ &\iff \operatorname{Ppart}(\langle \pi(g')\rangle) \in C \iff \alpha_C(\pi(g')) = 1\,. \end{aligned}$$

(b) follows from Lemma 3.2. Suppose that C_1 and C_2 are two different colorings of the Galois cover $V \overset{G}{\twoheadrightarrow} W$. Then there exists a conjugacy class $C \subset C_1 \setminus C_2$ (or vice versa), and the lemma yields a model K such that $X(V \overset{G}{\twoheadrightarrow} W, C_1)(K) \supsetneq X(V \overset{G}{\twoheadrightarrow} W, C_2)(K)$.

(3) Checking that χ_c is compatible with disjoint unions and with products is straight forward. (For the products, one uses a product property of the map χ_c of Lemma 3.3; cf. Lemma 8.7 of [9]).

We have to check condition $(*)$ of Theorem 1.1, i.e.

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)\,,$$

where all prime factors of $|G|$ lie in P. By Lemma 3.3, (3), it is enough to show that $\alpha_{\{1\}} = \frac{1}{|G|}\alpha_{\mathrm{reg}}$, where α_{reg} is the character of the regular representation of G. But indeed: $\alpha_{\{1\}}(g) = 1$ if $\operatorname{Ppart}(\langle g\rangle) = \{1\}$ and $\alpha_{\{1\}}(g) = 0$ otherwise. As all prime factors of $|G|$ lie in P, we have $\operatorname{Ppart}(\langle g\rangle) = \langle g\rangle$, so $\alpha_{\{1\}}(g) = 1$ only if $g = 1$.

The last property to prove is invariance under definable bijections. We do this by first reducing the problem several times, until we are in the situation of Lemma 3.1.

- A definable bijection $\phi\colon X \to X'$ also yields bijections to the graph of ϕ, so we may suppose that the map $X \to X'$ is the restriction of a projection (which we also denote by ϕ).
- We may suppose $X = X(V \overset{G}{\twoheadrightarrow} W, C)$ by treating each component of X separately. (Replace X' by the image of that component.)
- Next we may suppose $X' = X(V' \overset{G'}{\twoheadrightarrow} W', C')$ by treating each component of X' separately. One easily checks that the new preimage X is still defined by a single Galois cover. (Note that for this, the order of this and the previous step is important.)
- Using the density statements of Lemma 3.2 and Noetherian induction, we may suppose that the map $\phi\colon W \to W'$ is finite and étale. By refining the Galois covers, we may suppose $V = V'$.

We now have the following diagram:

$$\begin{tikzcd} V \arrow[d, two heads, "G"'] \arrow[dr, two heads, "G'"] & \\ W \arrow[r, two heads, "\phi"'] & W' \end{tikzcd}$$

By decomposing once more and using Lemma 3.3, we may suppose that C consists of a single conjugacy class of subgroups of G and $C' = C^{G'}$ is the induced class in G'. Moreover, we get $\frac{|C|}{|G|} = \frac{|C'|}{|G'|}$, as by assumption ϕ induces a bijection $X(V \overset{G}{\twoheadrightarrow} W, C)(K) \to X(V \overset{G'}{\twoheadrightarrow} W', C')(K)$. (Choose K using Lemma 3.2 such that $X(V \overset{G'}{\twoheadrightarrow} W', C')(K)$ is not empty.)

We want to show $\chi_c(V \overset{G}{\twoheadrightarrow} W, C) = \chi_c(V \overset{G'}{\twoheadrightarrow} W', C')$. By Lemma 3.3 (5), it is enough to show that $\alpha_{C'} = \operatorname{Ind}_G^{G'} \alpha_C$.

Set

$$\begin{aligned} \hat{C} &:= \{\langle g\rangle \subset G \mid \alpha_C(g) = 1\} = \{\langle g\rangle \subset G \mid \operatorname{Ppart}(\langle g\rangle) \in C\} \quad \text{and} \\ \hat{C}' &:= \{\langle g'\rangle \subset G' \mid \alpha_{C'}(g') = 1\} = \{\langle g'\rangle \subset G \mid \operatorname{Ppart}(\langle g'\rangle) \in C'\}\,. \end{aligned}$$

We want to understand the relation between $\hat{C}$ and $\hat{C}'$. For this, consider the map $\eta\colon \hat{C}' \to C', Q \mapsto \operatorname{Ppart}(Q)$. It maps $\hat{C}$ to C. We claim that $\hat{C}$ is exactly the preimage of C under η. For this, we have to verify that for any group $Q \in \hat{C}'$ with $\operatorname{Ppart}(Q) \in C$, we already have $Q \subset G$. Indeed: Q is abelian, so it is contained in $\mathrm{N}_{G'}(\operatorname{Ppart}(Q))$, and $\mathrm{N}_{G'}(\operatorname{Ppart}(Q))$ is contained in G.

Now using that C consists of a single conjugacy class and that η commutes with conjugation, we arrive at two conclusions: $\hat{C}' = \hat{C}^{G'}$ and $\frac{|\hat{C}'|}{|C'|} =$ fiber size of $\eta = \frac{|\hat{C}|}{|C|}$.

Using this, we can finally compute $\operatorname{Ind}_G^{G'} \alpha_C$. For any $g' \in G'$, we have

$$\operatorname{Ind}_G^{G'} \alpha_C(g') = \frac{1}{|G|}\#\{h \in G' \mid \langle hg'h^{-1}\rangle \in \hat{C}\}\,.$$

This is zero if $\langle g'\rangle \notin \hat{C}^{G'} = \hat{C}'$. Otherwise:

$$\cdots = \frac{1}{|G|} \cdot |\hat{C}| \cdot |\,\mathrm{N}_{G'}(\langle g'\rangle)| = \frac{|\hat{C}|}{|G|} \cdot \frac{|G'|}{|\hat{C}'|} = 1\,.$$

(In the last equality, we combine $\frac{|C|}{|G|} = \frac{|C'|}{|G'|}$ and $\frac{|\hat{C}'|}{|C'|} = \frac{|\hat{C}|}{|C|}$.)

3.3. **The uniqueness statement.** We now prove the uniqueness of the map χ_c in the case of pseudo-finite fields. For this, we only need following properties of χ_c: it extends the usual map $\chi_c\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Mot}_k)$, it is invariant under definable bijections, it is compatible with disjoint unions, and for any Galois cover $V \overset{G}{\twoheadrightarrow} W$, the equality

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)$$

holds.

In particular, we will not need that χ_c is compatible with products.

Proof of uniqueness in Theorem 1.1. By Lemma 2.10 (quantifier elimination to Galois formulas) and compatibility with disjoint unions, it is enough to prove uniqueness for definable sets of the form $X(V \overset{G}{\twoheadrightarrow} W, C)$, where $(V \overset{G}{\twoheadrightarrow} W, C)$ is a colored Galois cover and $C = Q^G$ consists of a single conjugacy class of cyclic subgroups of G.

We proceed by induction on $|G|$ and $|Q|$. (We will suppose that the statement is true for G of the same size and Q smaller and vice versa.)

Suppose first that Q is not normal in G. Let $G' := \mathrm{N}_G(Q)$ be its normalizer and $W' := V/G'$. Note that $C' := Q^{G'} = \{Q\}$. By induction, we know $\chi_c(X(V \overset{G'}{\twoheadrightarrow} W', C'))$. We have $\frac{|G'|}{|C'|} = \frac{|G|}{|C|}$, so Lemma 3.1 implies that the map $W' \to W$ induces a bijection $X(V \overset{G'}{\twoheadrightarrow} W', C') \to X(V \overset{G}{\twoheadrightarrow} W, C)$. So by assumption $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, C)) = \chi_c(X(V \overset{G'}{\twoheadrightarrow} W', C'))$.

Now suppose Q is normal in G (and in particular $C = \{Q\}$). Let $G' := G/Q$ and $V' := V/Q$. We know $\chi_c(X(V' \overset{G'}{\twoheadrightarrow} W, \{1\}))$ by $(*)$, and we have $X(V' \overset{G'}{\twoheadrightarrow} W, \{1\}) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where $C_1 = \{Q_1 \in \mathrm{Psub}(G) \mid Q_1 \subset Q\}$ consists of all (cyclic) subgroups of G contained in Q. But for any strict subgroup $Q_1 \subsetneq Q$, we know $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, Q_1^G))$ by induction. So $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, Q))$ is the only (up to now) unknown term in the equation

$$\chi_c(X(V \overset{G}{\twoheadrightarrow} W, C_1)) = \sum_{\substack{C_2 \subset C_1 \\ C_2 \text{ one conjugacy class}}} \chi_c(X(V \overset{G}{\twoheadrightarrow} W, C_2))\,.$$

□

4. Maps between Grothendieck rings

In this section we first prove the existence of the map θ_ι between the different Grothendieck rings $\mathrm{K}_0(T_{Gal,k})$ (Theorem 1.3) and then apply this to get Proposition 1.4. Finally we check a compatibility between the maps θ_ι and the maps χ_c.

4.1. **Existence of the maps θ_ι.** Recall the statement of the theorem. We have a field k and an inclusion of torsion-free pro-cyclic groups $\iota\colon Gal_2 \hookrightarrow Gal_1$. For simplicity, we will now identify Gal_2 with $\iota(Gal_2) \subset Gal_1$. Denote by $T_i := T_{Gal_i,k}$ the theory of perfect PAC fields with Galois group Gal_i and which contain k.

The map $\theta := \theta_\iota\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$ was defined as follows. Any model K_1 of T_1 yields a model $K_2 := \tilde{K}_1^{Gal_2}$ of T_2. For any $X_2 \subset \mathbb{A}^n$ definable in T_2, we defined $\theta(X_2)(K_1) = X_2(K_2) \cap K_1^n$.

What we have to check is:

1. $X_2(K_2) \cap K_1^n$ is definable (uniformly for all K_1).
2. If there is a definable bijection $X_2 \to X_2'$ in T_2, then there is also a definable bijection $\theta(X_2) \to \theta(X_2')$ in T_1.
3. θ is a ring homomorphism, i.e. compatible with disjoint unions and products.

The third statement is clear by definition.

(1) Any definable set X_2 of T_2 can be written as disjoint union of sets of the form $X(f\colon V \overset{G}{\twoheadrightarrow} W, C_2)$, where C_2 is a conjugacy class of permitted subgroups of G, so it is enough to prove that θ maps such sets to definable ones. We claim: $\theta(X(V \overset{G}{\twoheadrightarrow} W, C_2)) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where C_1 is defined as follows: Let M be the set of homomorphisms $\rho_1\colon\ Gal_1 \to G$ such that $\rho_1(Gal_2) \in C_2$. Then C_1 is the set of images of these homomorphisms M. In a formula:

$$C_1 = \{\operatorname{im}\rho_1 \mid \rho_1\colon\ \mathrm{Gal}_1 \to G, \rho_1(\mathrm{Gal}_2) \in C_2\}\,.$$

We have to check: For any model K_1 of T_1 and any element $w \in W(K_1)$, we have $w \in X(V \overset{G}{\twoheadrightarrow} W, C_1)(K_1)$ if and only if $w \in X(V \overset{G}{\twoheadrightarrow} W, C_2)(K_2)$, where $K_2 = \tilde{K}_1^{Gal_2}$ as above.

Choose an element $v \in V(\tilde{K_1})$ with $f(v) = w$. We get a homomorphism $\rho_1\colon\ Gal_1 \to G$ defined by $\sigma(v) = v.\rho_1(\sigma)$ for any $\sigma \in Gal_1$. Of course the restriction $\rho_2 := \rho_1|_{Gal_2}$ satisfies the same property. By definition, we have $w \in X(V \overset{G}{\twoheadrightarrow} W, C_1)(K_1)$ if and only if $\operatorname{im}\rho_1 \in C_1$ and $w \in X(V \overset{G}{\twoheadrightarrow} W, C_2)(K_2)$ if and only if $\operatorname{im}\rho_2 = \rho_1(\mathrm{Gal}_2) \in C_2$. So we have to check that for any $\rho_1\colon\ Gal_1 \to G$ we have $\operatorname{im}\rho_1 \in C_1$ if and only if $\rho_1(\mathrm{Gal}_2) \in C_2$.

"$\Leftarrow$" is clear by the definition of C_1.

"$\Rightarrow$": Suppose $Q_1 := \operatorname{im}\rho_1 \in C_1$. By the definition of C_1, there is a homomorphism $\rho_1' \in M$ with $\operatorname{im}\rho_1' = Q_1$. As Gal_1 is pro-cyclic, homomorphisms $Gal_1 \to Q_1$ are determined by the image of a generator, so we can write $\rho_1 = \alpha \circ \rho_1'$ for some automorphism $\alpha \in \operatorname{Aut}(Q_1)$. As Q_1 is cyclic, all its subgroups are characteristic subgroups, so $\rho_1(Gal_2) = \alpha(\rho_1'(Gal_2)) = \rho_1'(Gal_2) \in C_2$. This implies $\rho_1 \in C_1$.

(2) Suppose $X_2 \subset \mathbb{A}^n$ and $X_2' \subset \mathbb{A}^{n'}$ are two definable sets in T_2 and $f\colon X_2 \to X_2'$ is a definable bijection. We have to show that there is a T_1-definable bijection $\theta(X_2) \to \theta(X_2')$. Indeed, we will check that $\theta(f)$ is such a bijection. In other words we have to verify the following statement:

Let K_1 be any model of T_1 and $K_2 = \tilde{K}_1^{Gal_2}$. Then for any $x \in X_2(K_2)$ and $x' := f(x) \in X_2'(K_2)$, we have $x \in K_1^n$ if and only if $x' \in K_1^{n'}$.

Suppose $x \notin K_1^n$. Then there exists a $\sigma \in \mathrm{Gal}(K_2/K_1)$ moving x. But $\sigma(X_2(K_2)) = X_2(K_2)$, so $\sigma(x) \in X_2$. As f is injective on $X_2(K_2)$, this implies $\sigma(f(x)) = f(\sigma(x)) \neq f(x)$, so $f(x) \notin K_1^{n'}$.

The other direction works analogously. □

4.2. χ_c **is not injective.** As an example application of the maps θ_ι, we will now prove Proposition 1.4. To this end, we will construct a pair of definable sets X_1 and X_2 such that $\chi_c(X_1) = \chi_c(X_2)$ but $\chi_c(\theta_\iota(X_1)) \neq \chi_c(\theta_\iota(X_2))$ for a suitable map $\iota\colon\ Gal \hookrightarrow Gal$. (In fact, we will construct a whole bunch of such pairs.)

Proof of Proposition 1.4. Recall that Gal is a non-trivial subgroup of $\hat{\mathbb{Z}}$, i.e. $Gal = \prod_{p\in P} \mathbb{Z}_p$, where P is a non-empty set of primes.

For $n \in \mathbb{N}_{\geq 1}$, consider the group homomorphism $\iota\colon\ Gal \hookrightarrow Gal, \sigma \mapsto \sigma^n$. Applying Theorem 1.3 to this map gives an endomorphism θ_n of $\mathrm{K}_0(T_{\hat{\mathbb{Z}},k})$, which can be explicitly computed on sets defined by Galois covers as follows. Let $(V \overset{G}{\twoheadrightarrow} W, C_2)$ be a colored Galois cover. The computation in the proof of Theorem 1.3 shows that $\theta_n(X(V \overset{G}{\twoheadrightarrow} W, C_2)) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where $C_1 = \{Q \in \mathrm{Psub}(G) \mid Q^n \in C_2\}$ consists of those permitted subgroups of G whose subgroups of n-th powers lie in C_2.

Note that θ_n is interesting only if n has prime factors which lie in P; otherwise, n and $|Q|$ are coprime for any permitted subgroup $Q \subset G$, which implies $Q = Q^n$, $C_1 = C_2$, and $\theta_n = \mathrm{id}$.

Now let $V \overset{G}{\twoheadrightarrow} W$ be any non-trivial Galois cover such that all prime factors of $|G|$ lie in P, and define $X := X(V \overset{G}{\twoheadrightarrow} W, \{\mathrm{id}\})$. By condition $(*)$ of Theorem 1.1, we have $\chi_c(X) = \frac{1}{|G|}\chi_c(V)$, so $\chi_c(X \times G) = \chi_c(V)$. (Here G is interpreted as a discrete set.) However, we will see that for $n = |G|$, we have $\chi_c(\theta_n(X \times G)) \neq \chi_c(\theta_n(V))$.

As θ_n is the identity on $\mathrm{K}_0(\mathrm{Var}_k)$, we have $\theta_n(V) = [V]$. On the other hand, the subgroup of n-th powers of any cyclic subgroup of G is trivial, so $\theta_n(X) = [X(V \overset{G}{\twoheadrightarrow} W, \mathrm{Psub}(G))] = [W]$ and $\theta_n(X \times G) = [W \times G]$. But V and $W \times G$ are two varieties with a different number of irreducible components of maximal dimension, so $\chi_c(\theta_n(X \times G)) \neq \chi_c(\theta_n(V))$. □

4.3. **Compatibility of χ_c and θ_ι.** We prove the following compatibility statement:

Proposition 4.1. *Suppose k is a field of characteristic zero and $Gal_2 \subset Gal_1$ are two torsion-free pro-cyclic groups such that $Gal_1 \,/\, Gal_2$ is torsion-free, too. We use the following notation: $T_i := T_{Gal_i,k}$ (for $i = 1, 2$) are the corresponding theories, $\chi_c^i\colon \mathrm{K}_0(T_i) \to \mathrm{K}_0(\mathrm{Mot}_k)_{\mathbb{Q}}$ are the maps of Theorem 1.1, and $\theta\colon \mathrm{K}_0(T_2) \to \mathrm{K}_0(T_1)$ is the map provided by Theorem 1.3 applied to the inclusion $Gal_2 \subset Gal_1$. Then we have:*

$$\chi_c^2 = \chi_c^1 \circ \theta\,.$$

Proof. For $i = 1, 2$ let P_i be the set of primes such that $Gal_i = \prod_{p \in P_i} \mathbb{Z}_p$. We have $P_2 \subset P_1$, and Gal_2 is just the factor of Gal_1 corresponding to P_2. We will write Psub_i resp. Ppart_i for the permitted subgroups and the permitted part to distinguish between the two Galois groups.

We only have to verify the statement for sets of the form $X(V \overset{G}{\twoheadrightarrow} W, C_2)$, where $(V \overset{G}{\twoheadrightarrow} W, C_2)$ is a colored Galois cover for T_2. By the proof of Theorem 1.3, we have $\theta(X(V \overset{G}{\twoheadrightarrow} W, C_2)) = X(V \overset{G}{\twoheadrightarrow} W, C_1)$, where C_1 consists of the images of those maps $\rho\colon\ Gal_1 \to G$ which satisfy $\rho(Gal_2) \in C_2$. As Gal_2 is a direct factor of Gal_1, we get $C_1 = \{Q \in \mathrm{Psub}_1(G) \mid \mathrm{Ppart}_2(Q) \in C_2\}$.

Now recall the definition of χ_c^i: $\chi_c^i(X(V \overset{G}{\twoheadrightarrow} W, C_i)) = \chi_c(G \circlearrowright V, \alpha_{C_i})$, where

$$\alpha_{C_i}(g) := \begin{cases} 1 & \text{if } \mathrm{Ppart}_i(\langle g\rangle) \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

But $\mathrm{Ppart}_1(\langle g\rangle) \in C_1$ if and only if $\mathrm{Ppart}_2(\mathrm{Ppart}_1(\langle g\rangle)) = \mathrm{Ppart}_2(\langle g\rangle) \in C_2$, so $\alpha_{C_1} = \alpha_{C_2}$, and the claim is proven. □

5. Open problems

5.1. **Uniqueness of χ_c.** In the case of pseudo-finite fields, the conditions given in Theorem 1.1 are enough to render χ_c unique. One would like to have a similar uniqueness statement in the other cases. Unfortunately, the condition

$$(*) \qquad \chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \frac{1}{|G|}\chi_c(V)$$

is false in general if $|G|$ has prime factors not in P (where $Gal = \prod_{p \in P} \mathbb{Z}_p$). For algebraically closed fields for example, we have $\chi_c(X(V \overset{G}{\twoheadrightarrow} W, \{1\})) = \chi_c(W)$, which is not equal to $\frac{1}{|G|}\chi_c(V)$ unless G is trivial.

The first question is: is the weak version of $(*)$ (when one requires all prime factors of $|G|$ to lie in P) enough to get uniqueness? And if not: is there some other nice condition rendering χ_c unique? One fact suggesting that the weak condition might already be strong enough is that this is true indeed for algebraically closed fields.

5.2. **From motives to measure.** The parallels between the definitions of the virtual motive associated to a definable set and the measure of such a set ([2], [7]) suggest that one should be able to extract the measure from the motive. More precisely, fix a perfect PAC field K of characteristic zero with pro-cyclic Galois group Gal. Note that there are two theories around now: $T_{Gal,K}$, the theory of pseudo-finite fields containing K (which is not complete) and $\mathrm{Th}(K)$, the (complete) theory of K itself.

Denote by $\dim\colon \mathrm{K}_0(\mathrm{Th}(K)) \to \mathbb{N}$ the dimension of [3] (which needs not coincide with the usual dimension for varieties: only components "visible over K" are considered) and by $\mu\colon \mathrm{K}_0(\mathrm{Th}(K)) \to \mathbb{Q}$ the measure of [7]. The question is whether a dotted map in the following diagram exists making the diagram commutative.

$$\begin{tikzcd}
\mathrm{K}_0(T_K) \arrow[r, two heads] \arrow[d, "\chi_c"] & \mathrm{K}_0(\mathrm{Th}(K)) \arrow[d, "(\dim, \mu)"] \\
\mathrm{K}_0(\mathrm{Mot}_K)_{\mathbb{Q}} \arrow[r, dashed] & \mathbb{N} \times \mathbb{Q}
\end{tikzcd}$$

If K is algebraically closed, then this is obviously true: In this case $\mu(V)$ is just the number of irreducible components of maximal dimension of V, and both this and the dimension of V (which is the usual one in this case) can be seen in the corresponding motive.

If K is pseudo-finite, this is true, too: Let X be a definable set of $T_{Gal,K}$. Then it makes sense to speak about $X(F)$ for finite fields F of almost all characteristics. Lemma 3.3.2 of [4] states that for almost all characteristics, the number of points $|X(F)|$ is encoded in the motive. (Not very surprisingly, it is the trace of the Frobenius automorphism on the motive.) The dimension and the measure of X in K can be computed from these cardinalities.

The way one extracts the dimension and the measure from the motive seems quite different in the two above cases. This suggests that one might get interesting new insights by generalizing this to arbitrary pro-cyclic Galois groups.

5.3. **Larger Galois groups for the maps θ_ι.** The quantifier elimination result of [6] does not only work for fields with pro-cyclic Galois groups, but for some larger Galois groups as well. (The Galois group has to satisfy what Fried-Jarden call the "embedding property".) It seems plausible that Theorem 1.3 should be

generalizable to this context as well. However the proof will need some modifications. Indeed for $Gal_1 = \hat{\mathbb{Z}} * \hat{\mathbb{Z}} = \langle a, b\rangle$ and $Gal_2 = \langle a\rangle \subset Gal_1$, one can construct a T_2-definable set $X = X(V \overset{G}{\twoheadrightarrow} W, C)$ such that $\theta(X)$ is not definable using the same Galois cover $V \overset{G}{\twoheadrightarrow} W$.

5.4. **Larger Galois groups for the maps χ_c.** Another natural question is whether the map χ_c can also be defined for fields with larger Galois group. However, in [7] we already showed that the measure of [2] does not extend to this generality. Indeed, no measure exists for example if the Galois group is $\hat{\mathbb{Z}} * \hat{\mathbb{Z}}$. This suggests that it is neither possible to associate motives to definable sets of such theories. Probably, $T_{\hat{\mathbb{Z}}*\hat{\mathbb{Z}},k}$ contains too many definable bijections so that the corresponding Grothendieck ring gets too small. One might even hope to show that $\mathrm{K}_0(T_{\hat{\mathbb{Z}}*\hat{\mathbb{Z}},k})$ is trivial.

5.5. **What exactly do we know about $\mathrm{K}_0(T_{Gal,k})$?** We showed that the maps χ_c do not yield the full information about the definable sets and we showed how additional information can be obtained using the maps θ_ι. The question is now: how much information do we get using all maps θ_ι? More precisely, suppose X_1 and X_2 are two definable sets in $T_{Gal,k}$, and suppose that for any injective endomorphism $\iota\colon Gal \hookrightarrow Gal$ we have $\chi_c(\theta_\iota(X_1)) = \chi_c(\theta_\iota(X_2))$. What does this tell us about X_1 and X_2 (as elements of $\mathrm{K}_0(T_{Gal,k})$)?

The best we could hope would be $[X_1] = [X_2]$, but this is wrong in the case $Gal = \{1\}$: There are no non-trivial maps θ_ι, and the map $\chi_c\colon \mathrm{K}_0(T_{Gal,k}) \to \mathrm{K}_0(\mathrm{Mot}_k)$ is known to be non-injective for algebraically closed fields.

So what one could really hope for would be that "apart from this", the maps $\chi_c \circ \theta_\iota$ yield all additive information about the definable sets of $T_{Gal,k}$. The first open problem here is to give a precise meaning to this statement.

References

[1] S. del Baño Rollin & V. Navarro Aznar, "On the motive of a quotient variety", *Collect. Math.* **49** (1998), no. 2-3, p. 203-226, Dedicated to the memory of Fernando Serrano.

[2] Z. Chatzidakis, L. van den Dries & A. Macintyre, "Definable sets over finite fields", *J. Reine Angew. Math.* **427** (1992), p. 107-135.

[3] Z. Chatzidakis & E. Hrushovski, "Perfect pseudo-algebraically closed fields are algebraically bounded", *J. Algebra* **271** (2004), no. 2, p. 627-637.

[4] J. Denef & F. Loeser, "Definable sets, motives and p-adic integrals", *J. Amer. Math. Soc.* **14** (2001), no. 2, p. 429-469 (electronic).

[5] ———, "On some rational generating series occurring in arithmetic geometry", in *Geometric aspects of Dwork theory. Vol. I, II*, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, p. 509-526.

[6] M. D. Fried & M. Jarden, *Field arithmetic*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 11, Springer-Verlag, Berlin, 2005, xxiv+780 pages.

[7] I. Halupczok, "A measure for perfect PAC fields with pro-cyclic Galois group.", *J. Algebra* (2007), no. 310, p. 371-395.

[8] E. Hrushovski, "Pseudo-finite fields and related structures", in *Model theory and applications*, Quad. Mat., vol. 11, Aracne, Rome, 2002, p. 151-212.

[9] J. Nicaise, *Relative motives and the theory of pseudo-finite fields*, Int. Math. Res. Pap., vol. 2007:rpm001, 2007, 69 page.

DMA, Ecole Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05 – France
E-mail address: math@karimmi.de